
NEURAL NETWORKS

Hyperbolic tangent functions as neuronal signals

Solve the following five problems.

Problem 1

Consider a neuronal signal of the form of a hyperbolic tangent function $h(\alpha x) = \tanh(\alpha x)$, where α is a positive parameter. Verify that

$$(a) h(\alpha x) = \frac{2}{1 + e^{-2\alpha x}} - 1$$

$$(b) \frac{d}{dx} [h(\alpha x)] = \alpha [1 - h^2(\alpha x)]$$

$$(c) \frac{d}{dx} [h(\alpha x)] \bigg|_{\max} = \alpha, \text{ occurring at } x = 0$$

$$(d) \frac{d^2}{dx^2} [h(\alpha x)] = -2\alpha^2 h(\alpha x) [1 - h^2(\alpha x)]$$

Problem 2

A neuron receives two inputs $x_1 = 1.3$ and $x_2 = 1.5$ with inputs $w_1 = 1$ and $w_2 = 2$, respectively. The bias weight w_0 is a design parameter. The neuron employs a hyperbolic tangent function of the form $s = \tanh(0.8y)$, where s is the output signal and y is the activation.

(a) For $w_0 = -2.5$, find s .

(b) For $s = 0.71$, find w_0 .

Problem 3

A single neuron receives two inputs $x_1 = 1.5$ and

$x_1 = 2$ with weights $w_1 = 2$ and $w_2 = -1.1$, respectively. The bias weight is $w_0 = 0.4$. The neuron employs a hyperbolic tangent function of the form $s = \tanh(0.75y)$, where s is the output signal and y is the activation.

- Find the value of s .
- Find the value of the derivative $\frac{ds}{dy}$.
- Plot s as a function of y and locate the operating point (y, s) .
- Plot $\frac{ds}{dy}$ as a function of y and locate the operating point $(y, \frac{ds}{dy})$.
- What is the maximum value of $\frac{ds}{dy}$? Where does it occur?

Problem 4

A single neuron receives three inputs $x_1 = 2$, $x_2 = 3$, and $x_3 = 2.5$ with weights $w_1 = 1.2$, $w_2 = -1.1$, and $w_3 = -1$, respectively. The bias weight w_0 is a design parameter. The neuron employs a hyperbolic tangent function of the form $s = \tanh(0.5y)$, where s is the output signal and y is the activation. Find the value of w_0 such that $\frac{ds}{dy} = 0.226$. What are the corresponding values of y and s ?

Problem 5

A single neuron receives two inputs $x_1 = 0.8$ and $x_2 = 1.2$ with weights $w_1 = 1.6$ and $w_2 = 0.6$, respectively. The bias weight is $w_0 = -1.4$. The neuron employs a hyperbolic tangent function of the form $s = \tanh(\alpha y)$, where s is the output signal, y is the activation, and α is a positive parameter. The derivative of s with respect to y is found to be 0.311. Calculate the values of α , y , and s .

Consider a neuronal signal of the form of a hyperbolic tangent function $h(\alpha x) = \tanh(\alpha x)$, where α is a positive parameter. Verify that

$$(a) h(\alpha x) = \frac{2}{1 + e^{-2\alpha x}} - 1$$

$$(b) \frac{d}{dx} [h(\alpha x)] = \alpha [1 - h^2(\alpha x)]$$

$$(c) \frac{d}{dx} [h(\alpha x)] \Big|_{\max} = \alpha, \text{ occurring at } x = 0$$

$$(d) \frac{d^2}{dx^2} [h(\alpha x)] = -2\alpha^2 h(\alpha x) [1 - h^2(\alpha x)]$$

Solution

$$\begin{aligned} (a) \text{L.H.S.} &= \tanh(\alpha x) = \frac{\sinh(\alpha x)}{\cosh(\alpha x)} = \frac{(e^{\alpha x} - e^{-\alpha x})/2}{(e^{\alpha x} + e^{-\alpha x})/2} \\ &= \frac{1 - e^{-2\alpha x}}{1 + e^{-2\alpha x}} = \frac{2 - 1 - e^{-2\alpha x}}{1 + e^{-2\alpha x}} = \frac{2}{1 + e^{-2\alpha x}} - 1 \\ &= \text{R.H.S.} \end{aligned}$$

$$\begin{aligned} (b) \frac{d}{dx} [\tanh(\alpha x)] &= \alpha \operatorname{sech}^2(\alpha x) = \alpha [1 - \tanh^2(\alpha x)] \\ &= \alpha [1 - h^2(\alpha x)] \end{aligned}$$

(c) To obtain the maximum value of the first derivative $\frac{d}{dx} [h(\alpha x)]$, we find an expression for the second derivative $\frac{d^2}{dx^2} [h(\alpha x)]$ and equate it to zero.

$$\begin{aligned} \frac{d^2}{dx^2} [\tanh(\alpha x)] &= \frac{d^2}{dx^2} [\alpha \operatorname{sech}^2(\alpha x)] \\ &= -2\alpha^2 \operatorname{sech}^2(\alpha x) \tanh(\alpha x) \\ &= 0 \end{aligned}$$

Since α and $\operatorname{sech}(\alpha x)$ are greater than zero, we

should have

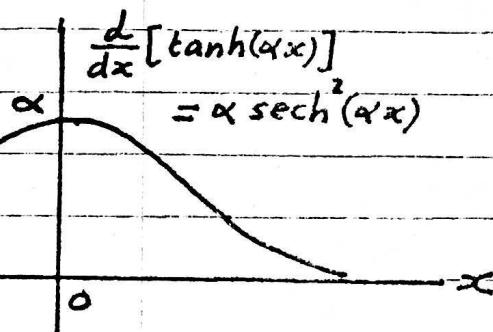
$$\tanh(\alpha x) = 0$$

or

$$x = 0$$

Therefore,

$$\frac{d}{dx} [\tanh(\alpha x)] \Big|_{\max} = \alpha [1 - \tanh^2(\alpha x)] \Big|_{x=0} = \infty$$



(d) From the solution of part (c),

$$\begin{aligned} \frac{d^2}{dx^2} [\tanh(\alpha x)] &= -2\alpha^2 \operatorname{sech}^2(\alpha x) \tanh(\alpha x) \\ &= -2\alpha^2 \tanh(\alpha x) [1 - \tanh^2(\alpha x)] \\ &= -2\alpha^2 h(\alpha x) [1 - h^2(\alpha x)] \end{aligned}$$

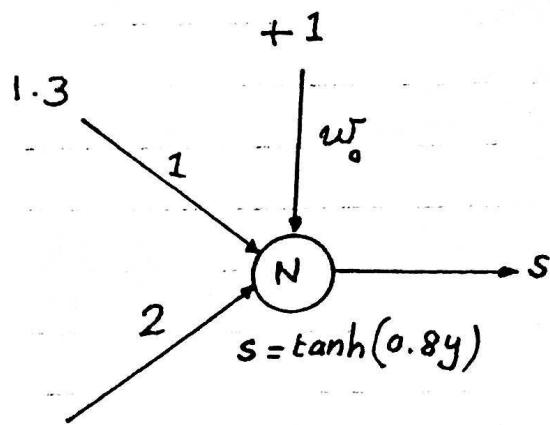
A neuron receives two inputs $x_1 = 1.3$ and $x_2 = 1.5$ with inputs $w_1 = 1$ and $w_2 = 2$, respectively. The bias weight w_0 is a design parameter. The neuron employs a hyperbolic tangent function of the form $s = \tanh(0.8y)$, where s is the output signal and y is the activation.

- For $w_0 = -2.5$, find s .
- For $s = 0.71$, find w_0 .

Solution

Activation,

$$\begin{aligned} y &= (1.3)(1) + (1.5)(2) + w_0 \\ &= 4.3 + w_0 \end{aligned}$$



- For $w_0 = -2.5$,

$$\begin{aligned} y &= 4.3 - 2.5 = 1.8 \\ s &= \tanh(0.8 \times 1.8) \\ &= \underline{0.894} \end{aligned}$$

- For $s = 0.71$,

$$\begin{aligned} \tanh(0.8y) &= 0.71 \\ 0.8y &= \tanh^{-1}(0.71) = 0.887 \\ y &= 1.109 \\ 4.3 + w_0 &= 1.109 \\ w_0 &= \underline{-3.191} \end{aligned}$$

single neuron receives two inputs $x_1 = 1.5$ and $x_2 = 2$ with weights $w_1 = 2$ and $w_2 = -1.1$, respectively.

The bias weight is $w_0 = 0.4$. The neuron employs a hyperbolic tangent function of the form $s = \tanh(0.75y)$, where s is the output signal and y is the activation.

- Find the value of s .
- Find the value of the derivative $\frac{ds}{dy}$.
- Plot s as a function of y and locate the operating point (y, s) .
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Solution

Activation,

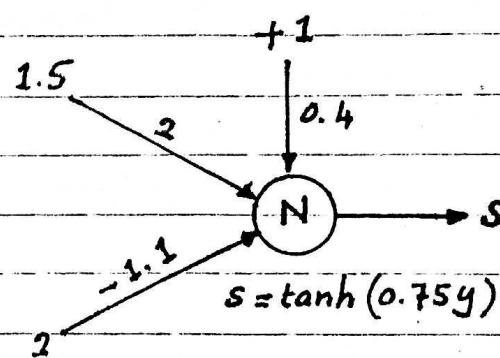
$$y = (1.5)(2) + (2)(-1.1) + 0.4 \\ = 1.2$$

Output signal,

$$s = \tanh(0.75 \times 1.2) = 0.716$$

Derivative,

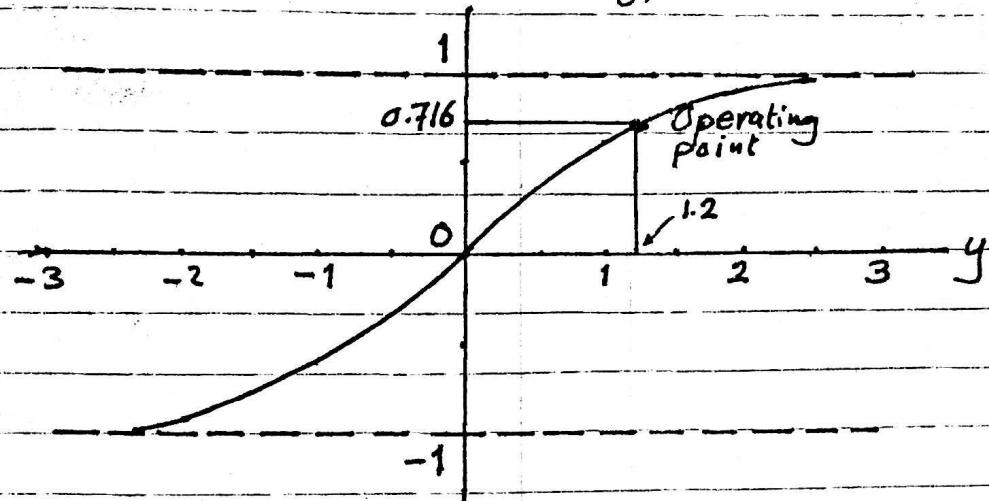
$$\frac{ds}{dy} = 0.75 \operatorname{sech}^2(0.75y) = 0.75 [1 - \tanh^2(0.75y)] \\ = 0.75 (1 - s^2) \\ = 0.366$$



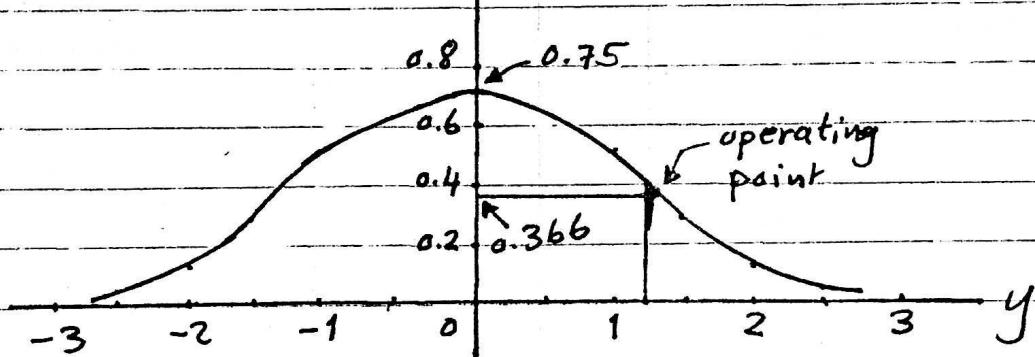
y	-2.5	-2	-1.5	-1	-0.5	0
$s = \tanh(0.75y)$	-0.954	-0.905	-0.809	-0.635	-0.358	0
$\frac{ds}{dy} = 0.75(1-s^2)$	0.067	0.136	0.259	0.448	0.654	0.75

	0.5	1	1.5	2	2.5
	0.358	0.635	0.809	0.905	0.954
	0.654	0.448	0.259	0.136	0.067

$$s = \tanh(0.75y)$$



$$\begin{aligned}\frac{ds}{dy} &= 0.75 [1 - \tanh^2(0.75y)] \\ &= 0.75 (1 - s^2)\end{aligned}$$



Maximum value of $\frac{dy}{ds} = 0.75$, and occurs at $y = 0$.

A single neuron receives three inputs $x_1 = 2$, $x_2 = 3$, and $x_3 = 2.5$ with weights $w_1 = 1.2$, $w_2 = -1.1$, and $w_3 = -1$, respectively. The bias weight w_0 is a design parameter. The neuron employs a hyperbolic tangent function of the form $s = \tanh(0.5y)$, where s is the output signal and y is the activation. Find the value of w_0 such that $\frac{ds}{dy} = 0.226$. What are the corresponding values of y and s ?

Solution

Activation,

$$\begin{aligned} y &= (2)(1.2) + (3)(-1.1) \\ &\quad + (2.5)(-1) + w_0 \\ &= -3.4 + w_0 \end{aligned}$$

Derivative,

$$\begin{aligned} \frac{ds}{dy} &= 0.5 \operatorname{sech}^2(0.5y) \\ &= 0.5 [1 - \tanh^2(0.5y)] \\ &= 0.226 \end{aligned}$$

$$\tanh^2(0.5y) = 0.548$$

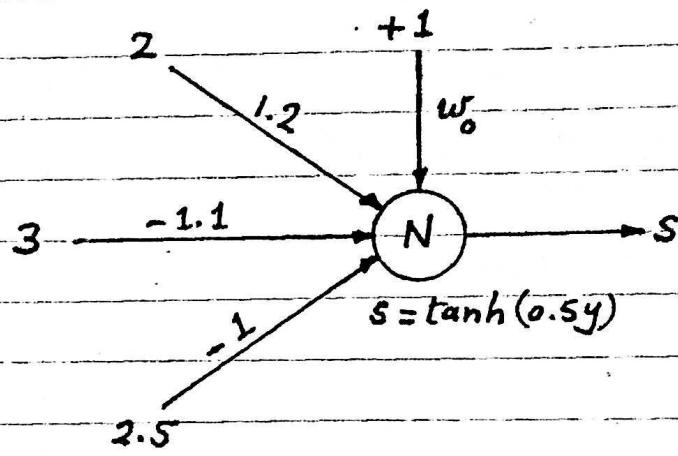
$$\tanh(0.5y) = \sqrt{0.548} = \pm 0.74$$

$$0.5y = \pm 0.95$$

$$y = \pm 1.9$$

$$\text{For } y = 1.9, w_0 = 1.9 + 3.4 = 5.3, s = \tanh(0.5 \times 1.9) = 0.74$$

$$\text{For } y = -1.9, w_0 = -1.9 + 3.4 = 1.5, s = \tanh(-0.5 \times 1.9) = -0.74$$



A single neuron receives two inputs $x_1 = 0.8$ and $x_2 = 1.2$ with weights $w_1 = 1.6$ and $w_2 = 0.6$, respectively. The bias weight is $w_0 = -1.4$. The neuron employs a hyperbolic tangent function of the form $s = \tanh(\alpha y)$, where s is the output signal, y is the activation, and α is a positive parameter. The derivative of s with respect to y is found to be 0.311. Calculate the values of α , y , and s .

Solution

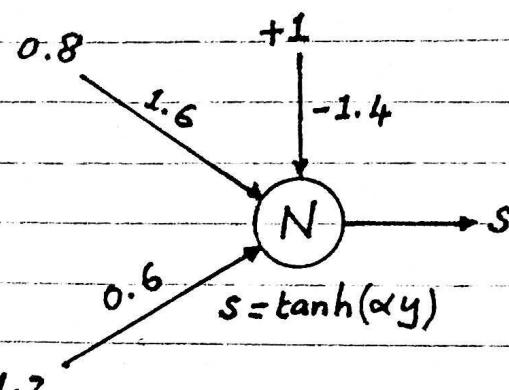
Activation,

$$y = (0.8)(1.6) + (1.2)(0.6) - 1.4 \\ = 0.6$$

Derivative,

$$\frac{ds}{dy} = \alpha \operatorname{sech}^2(\alpha y) \\ \frac{ds}{dy} = \alpha [1 - \tanh^2(\alpha y)] \\ = 0.311$$

$$\tanh^2(0.6\alpha) = 1 - \frac{0.311}{\alpha}$$



α	1	2	3	4	5	6	$\rightarrow \infty$
$\tanh^2(0.6\alpha)$	0.288	0.695	0.896	0.968	0.990	0.997	$\rightarrow 1$
$1 - \frac{0.311}{\alpha}$	0.689	0.845	0.896	0.922	0.938	0.948	$\rightarrow 1$

function

From the graph,

$$\alpha = 3$$

Output signal,

$$s = \tanh(0.6 \times 3)$$

$$= 0.947$$

